The analysis of the TR noise for a beam going through the substrate is analyzed comprehensively by Heinert *et. al.* [1]. The problem that we are looking at is the TE noise calculation for the same. Although, the algorithm for the same has been presented for the case of the laser beam reflecting off the surface, the calculation for the beam travelling through the substrate has not been considered as such.

Stress and Strain

The TE noise arises out of thermal fluctuations which result in random thermal expansion of the material generated by means of internal stresses and strains generated due to the temperature fluctuation. Most of the analysis is referred to the content in the text on Elastic Theory by Landau and Lifshitz[2]. Due to its importance, the expression related to the temperature field, $\vec{T}$, and the displacement field, $\vec{u}$, will be revised.

Displacement Field and Strain

The change in displacement field, $d\vec{u}$, is defined to be the change in position that two close points undergo when a deformation is applied to it. The strain tensor is defined based on the displacement field as Eq.(1.3) in [2]

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}\right) \tag{1}$$

However, the last term in the braket is small and neglected giving

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) \tag{2}$$

It follows that the change in volume is given by the sum of the elements of the diagonal elements of the tensor

$$dV' = dV\,(1 + u_{ii}) \tag{3}$$

The diagonal elements give the change in dimensions along the principal axes while the off-diagonal terms which do not alter the volume of the body are the shear strains.

Stress

The stress tensor is mathematically defined as the quantity, the divergence of which gives the force in any part of the body. So that, we have from the divergence theorem

$$\int F_i\,dV = \oint \sigma_{ik}\,df_k \tag{4}$$

The pressure acting on a surface represented by $\hat{n}$ is given as $\sigma_{ik}n_k = P_i$. In the special case of uniform pressure acting from all sides, also called hydrostatic compression, the expression for stress becomes $\sigma_{ik} = -p\delta_{ik}$

Thermodynamics

The basic equations of the four thermodynamic potentials change in the case when we are considering stress and strain. The internal energy follows

$$dE = TdS + \sigma_{ik} du_{ik} \tag{5}$$

In the special case of hydrostatic compression, the formula changes to the familiar form

$$dE = TdS - PdV \tag{6}$$

Using the definition of Helmholtz free energy, $F = E - TS$, we get

$$dF = -SdT + \sigma_{ik} du_{ik} \tag{7}$$

From Eq.(5) and Eq.(7) we get the following expression

$$\sigma_{ik} = \left(\frac{\partial E}{\partial u_{ik}}\right)_S = \left(\frac{\partial F}{\partial u_{ik}}\right)_T \tag{8}$$

Hooke's Law

The content of the Hooke's Law can be summed up as follows - The Free Energy is expanded only in second powers of u_{ik}. This gives a linear relation between the stress and the strain.

$$F = F_0 + \frac{1}{2}\lambda u_{ii}^2 + \mu u_{ik}^2 \tag{9}$$

where λ and μ are the Lame coefficients and can be expressed in terms of the Poisson ratio σ and Young's modulus Y. It is under these assumptions that one may write, the Free energy as

$$F = \frac{1}{2}\sigma_{ik} u_{ik} \tag{10}$$

One may also express the strain as

$$u_{ik} = \frac{\partial F}{\partial \sigma_{ik}} \tag{11}$$

under the assumption of the Hooke's Law.

Deformations with temperature change

In the consideration of thermal expansion, an extra term involving the thermal expansion coefficient, α, is added in the expansion of the Free energy, see Eq.(6.1) of [2].

$$F(T) = F_0(T) - K\alpha(T - T_0)u_{ll} + \mu(u_{ik} - \frac{1}{3}\delta_{ik} u_{ll})^2 + \frac{1}{2}K u_{ll}^2 \tag{12}$$

where T_0 is the equilibrium temperature of the body about which small changes occur. Now, in the adibatic case, if one uses the expression $S = -\frac{\partial F}{\partial T}$ and the fact that change in entropy $\Delta S = \frac{\Delta Q}{T} = \frac{C(T-T_0)}{T_0}$, one ends up with Eq.(6.5) of [2] which has been used in some of the TE noise calculations to relate the temperature perturbation from equilibrium to the expansion. The formula goes as

$$C(T - T_0)/T_0 = -K\alpha u_{ll} \tag{13}$$

where K is the bulk modulus which can be expressed in terms of Y and σ. Noting the fact that $u_{ll} = \partial u_l/\partial x_l = \nabla.\vec{u}$, one gets the expression used in Liu and Thorne[3] which relates the temperature perturbations with the expansion $\Theta = \nabla.\vec{u}$. It should be noted that while in [2] the α used is the coefficient of volume expansion, the same in the TE calculations uses the coefficient of linear expansion which is a factor of 3 small is magnitude.

Equation of Stress balance

The equation that considers the balancing of forces due to deformation is given as

$$\partial\sigma_{ik}/\partial x_k = F_i \tag{14}$$

Upon using the results of Hooke's law, according to which there is a linear relation between the stress and strain and considering behaviour of bodies in a gravitational field this reduces to Eq.(7.1) of [2]. In case where there is no force field but an external pressure being applied to one of the faces, this equation changes to its homogenous form

$$(1 - 2\sigma)\nabla^2\vec{u} + \nabla(\nabla.\vec{u}) = 0 \tag{15}$$

This is the equation solved for in TE noise calculation in [3] where the sinusoidal pressure in time appears at the boundary condition on the stress. In the presence of a heat source in the body, the free energy takes the form given in Eq.(12). The stress, consequently includes a term $-K\alpha(T - T_0)\delta_{ik}$ following Eq.(8). Hence, the equation of stress balance becomes Eq.(7.8) of [2]

$$\frac{3(1-\sigma)}{1+\sigma}\nabla(\nabla.\vec{u}_)) - \frac{3(1-2\sigma)}{2(1+\sigma)}\nabla \times \nabla \times \vec{u} = \alpha\nabla T \tag{16}$$

this equation is slightly different from the previous case due to the application of the identity $\nabla \times \nabla \times V \equiv \nabla(\nabla.V) - \nabla^2 V$. Thus, in the study of stresses generated due to temperature field, it is the above equation which is to be used.

Comments

- In is to be noted that the effect of the thermal stresses in a point inside the body will change the dimension of the body. Eq.(13) says that a temperature perturbation will affect the sum of the diagonal elements of the strain tensor, which is the relative change in volume and will not induce shear.

- In such a situation, considering the substrate, the net effect will be reflected on the outer surfaces.
- This fact enables us to follow the steps of Liu and Thorne[3] and apply a pressure on the two flat faces of the test mass and solve for the displacement field and evaluated the noise spectrum in an analogous manner.
- Other approaches could be to apply a heat source like Heinert[1] find out the temperature field and solve Eq.(16 instead for the displacent field and then evaluate the work dissipated as in [3]

References

[1] D. Heinert *et. al.* Phys. Rev. D **84**, 062001

[2] L.D. Landau and E.M. Lifshitz, *Theory of Elasticity*, third ed. (Pergamon Press Oxford, 1986)

[3] Yuk Tung Liu and Kip S. Thorne, Phys Rev. D **62**, 122002 (2000)