The "Work Dissipated"

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In the papers by Heinert and Liu and Thorne, the entire exercise hinges on the fact that the spectral density is related to the rate of dissipation of energy that is inject in by the heat source or by the pressure source. The expression being

$$S(\omega) = \frac{8k_BT}{F_0^2\omega^2}W_{diss}$$

The expression of W_{diss} used goes as

$$W_{diss} = \left\langle \frac{TdS}{dt} \right\rangle = \left\langle \int \frac{\kappa}{T} (\nabla T)^2 \, d^3r \right\rangle$$

However, this formula can be obtained from the heat equation itself. Consider the heat equation with $T(\vec{r,t})$ as the temperature field.

$$C_V \frac{\partial T(\vec{r},t)}{\partial t} + \kappa \nabla^2 T(\vec{r},t) = \dot{q}(\vec{r},t)$$

where $\dot{q}$ is the rate of heat flow. If one multiples the above equation with the temperture field T, one gets

$$C_V T \frac{\partial T}{\partial t} + \kappa T \nabla^2 T = T\dot{q} \tag{1}$$

Consider the first identity due to Green which says if one has two scalar field Φ and Ψ then the following holds

$$\int_V \left(\Phi\nabla^2\Psi + \nabla\Phi.\nabla\Psi\right) d^3r = \int_S \Phi\nabla\Psi.d\vec{S}$$

where V denotes the volume integral over any volume while S is the surface that closes that volume. Now consider both $\Phi = \Psi = T$ with the assumption that ∇T at the surface vanishes. This is the case that is considered in most papers (adiabatic boundary conditions). This implies

$$\int_V \left(T\nabla^2 T + \nabla T.\nabla T\right) d^3r = 0$$

$$\Rightarrow \int\limits_V T\nabla^2 T\, d^3r = -\int\limits_V (\nabla T)^2\, d^3r \tag{2}$$

Now consider the time average of the Eq.(1). The system being in equilibrium, the change in T with time i.e. $\frac{\partial T}{\partial t}$ will be small and will also time average to zero since the temperature fluctuation are random. Thus $\langle T\frac{\partial T}{\partial t}\rangle$ goes to zero. Applying Eq.(2), integrating over the volume Eq.(1) changes as

$$\left\langle -\int\limits_V \kappa\, (\nabla T)^2\, d^3r \right\rangle = T\left\langle \frac{dQ}{dt} \right\rangle$$

where $Q = \int q\, d^3r$. Since $dQ = TdS$ from thermodynamics, the above equation becomes

$$\left\langle -\int\limits_V \frac{\kappa}{T}\, (\nabla T)^2\, d^3r \right\rangle = \left\langle \frac{TdS}{dt} \right\rangle = W_{diss} \tag{3}$$

where T has been moved inside the integral since it hardly changes and is regarded a constant. This is the formula being used in the papers. The expression has an extra minus($-$) sign preceding the left hand side but apart from that the rest of it matches as given in Liu & Thorne. The minus sign has probably handled some way inside the average or maybe I have made an error.

A few points of discussion are as follows

- Liu and Thorne never considered the Heat equation but have used Eq.(3) for the dissipation, which is derived entirely by considering the Heat equation.
- An important to note it the fact that the above formula would not be true if $\nabla T \neq 0$ at the surface. This is what Heinert explicitly mentiones.
- It is evident that a heat source is to be used if Eq.(3) is to be used. However, Liu and Thorne had to see the surface effects - the way noise is added when beam reflects off the surface. So they applied pressure on the face so that the situation is correctly described and also one avoids injecting heat directly.
- Thus, in our simulation, the adiabatic boundary conditions are to be met all the time if we extract the gradient of temperature from COMSOL and use Eq.(3) in MATLAB to find the spectral density as in the codes of TR noise.