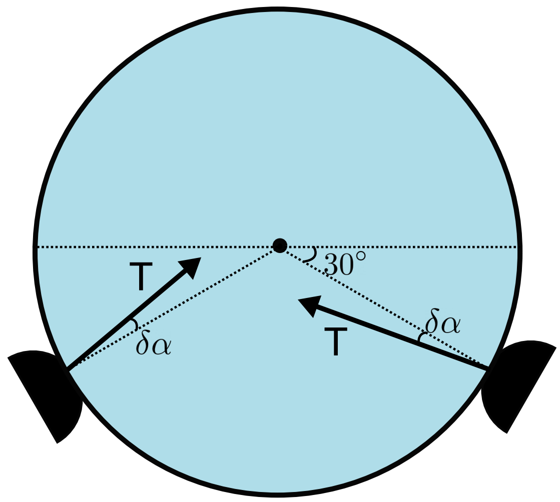
Coupling of Seismic Noise to Cavity Resonance Frequency due to mirror birefringence

In [1]:
```
from scipy.constants import c
import numpy as np
import matplotlib.pyplot as plt                  #For plotting
from matplotlib import cm, colors
figlist = []
#***********************************************************************************
#Setting RC Parameters for figure size and fontsizes
import matplotlib.pylab as pylab
params = {'figure.figsize': (16, 12),
          'xtick.labelsize':'xx-large',
          'ytick.labelsize':'xx-large',
          'text.usetex': False,
          'lines.linewidth': 4,
          'font.family': 'serif',
          'font.serif': 'Georgia',
          'font.size': 20,
          'xtick.direction': 'in',
          'ytick.direction': 'in',
          'xtick.labelsize': 'medium',
          'ytick.labelsize': 'medium',
          'axes.labelsize': 'medium',
          'axes.titlesize':'medium',
          'axes.grid.axis': 'both',
          'axes.grid.which': 'both',
          'axes.grid': True,
          'grid.color': 'xkcd:cement',
          'grid.alpha': 0.3,
          'lines.markersize': 12,
          'lines.linewidth': 2.0,
          'legend.borderpad': 0.2,
          'legend.fancybox': True,
          'legend.fontsize': 'medium',
          'legend.framealpha': 0.8,
          'legend.handletextpad': 0.5,
          'legend.labelspacing': 0.33,
          'legend.loc': 'best',
          'savefig.dpi': 140,
          'savefig.bbox': 'tight',
          'pdf.compression': 9}
pylab.rcParams.update(params)
#***********************************************************************************
from IPython.display import SVG, display, clear_output
```

In [2]:
```
# Some system constants and derived parameters
L = 1.45*25.4e-3                             # Length of cavity
La = 1.17*25.4e-3                            # Distance between points of contact.
Ro = 1.5*25.4e-3/2                           # Outer radius, m
Ri = 0.375*25.4e-3/2                         # Bore Radius, m
rho = 2202                                   # Density, kg/m^3
R_mir = 0.5*25.4e-3                          # Radius of end mirrors
d_mir = 0.25*25.4e-3                         # Thickness of end mirrors
M_mir = rho*np.pi*R_mir**2*d_mir                # Mass of each mirror
M = rho*np.pi*(Ro**2 - Ri**2)*L +2*M_mir     # Mass of cavity + mirros
I = rho*np.pi*La*(Ro**4 - Ri**4)/4           # Moment of Intertia along axis
J = np.pi * (Ro**4 - Ri**4)/2                # Second Polar Area moment
G = 31.2e9                                   # Shear Elasticity Modulus, Pa
xe = (L - La)/2                              # Distance of end point to point of cont
me = 0.5e-3                                  # Mounting position error
delta_alpha = me/Ro                          # Error in angle of Normal force from su
```



Schematic for one side of supports

Let's **assume** that the supports are slightly pressed on the cavity and misplaced such that the total normal force from the supports is not dieted exactly towards the center of the cavity. Further, this misalignment would have a common misalignment on both sides which would not result in a net torque and a differential misalignment. We call this differential misalignment $\delta alpha$, which is equal and opposite on the two sides. Force per seismic acceleration applied by each support on the cavity:

$$N\sin(30^{\circ}) = \frac{Ma_y}{4}$$
$$\frac{N}{a_y} = \frac{M}{4\sin(30^{\circ})}$$
$$T = \frac{M}{4\sin(30^{\circ})}$$

In [3]:
```
T = M/(4*np.sin(30*np.pi/180))
```

For $\delta\alpha$ error in angle of application of Normal force to the cavity, a resulting torque will act along the axis of the cavity (calculating torque per vertical seismic acceleration):

$$\mathcal{T} = 2R_oT\sin(\delta\alpha) \approx 2R_oT\delta\alpha$$

In [4]:
```
Tau = 2*Ro*T*delta_alpha
```

Assuming worst case scenario (or that $\delta\alpha$ is the differential error) so that the opposite points of support are all misaligned by this angle, the central plane of the cavity would be fixed while the two ends will twist in opposite direction. In this case, the twist angle at the Airy points (per vertical seismic acceleration) would be given by (Source:https://www.bu.edu/moss/mechanics-of-materials-torsion/ (https://www.bu.edu/moss/mechanics-of-materials-torsion/)):

$$\phi = \frac{\mathcal{T} L_A}{2GJ}$$

where L_A is the ditance between points of support on cavity, G is shear elasticity modulous for the spacer and J is second moment of area given by $\frac{\pi}{2}(R_o^4 - R_i^4)$. **Assuming that the rest of the cavity wouldn't twist anymore**, the difference of twist between the end mirror (per vertical seismic acceleration) is given by:

$$\Phi = 2 * \phi = \frac{\mathcal{T} L_A}{GJ}$$

$$\Phi = \frac{2R_o T \delta\alpha L_A}{G\frac{\pi}{2}(R_o^4 - R_i^4)}$$

$$= \frac{4R_o M \delta\alpha L_A}{4\sin(30^\circ) G\pi(R_o^4 - R_i^4)}$$

$$= \frac{2R_o M \delta\alpha L_A}{G\pi(R_o^4 - R_i^4)}$$

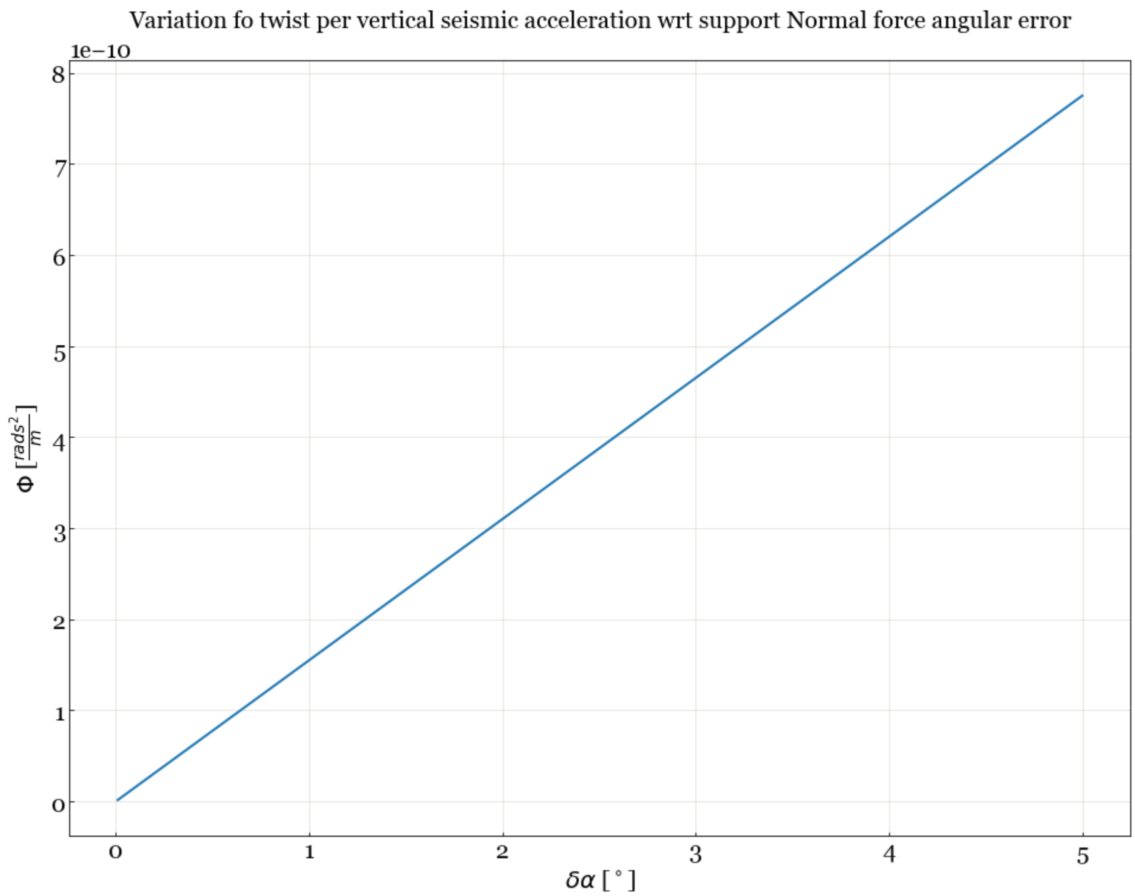
In [5]:
```
Phi = Tau*La/G/J
print('Twist angle between end mirrors per vertical seismic '
      + 'accelration for angular error of {:.2f} degrees is:'.format(delta_alpha*180
print("{:.2e} ".format(Phi) + r'[rad s^2/m]')
```

```
Twist angle between end mirrors per vertical seismic accelration for angular error
of 1.50 degrees is:
2.33e-10 [rad s^2/m]
```

Plotting twist per accelaration as function of support angular error $\delta\alpha$

In [6]:
```
da = np.linspace(0.01, 5, 500)      # Angular error in degrees
dar = da * np.pi / 180
Phiar = 2*Ro*M*La*dar/(G*np.pi*(Ro**4 - Ri**4))
fig = plt.figure()
ax = fig.gca()
ax.plot(da, Phiar)
ax.set_title('Variation fo twist per vertical seismic acceleration wrt support Norma
ax.set_xlabel(r'$\delta \alpha$ [$^\circ$]')
ax.set_ylabel(r'$\Phi$ [$\frac{rad s^2}{m}$]')
```

Out[6]:
```
Text(0, 0.5, '$\\Phi$ [$\\frac{rad s^2}{m}$]')
```



Effect on cavity resonant frequency

Let γ_1 be phase anisotropy of mirror 1 and γ_2 be phase anisotropy of mirror 2 and mirror 2's fast axis is at an angle ν with respect to mirror 1;s fast axis (which is assumed to be along x-axis). Then following the calculations of F. Brandi et al., J. Opt. Soc. Am. B 15, 1278 (1998) (https://link.springer.com/article/10.1007/s003400050283), this cavity is equivalent to a cavity with isotropic mirrors but a waveplate of phase anisotopy of γ_{EQ} at an angle ν_{EQ} with respect to x-axis. These are given by:

$$\gamma_{EQ} = \sqrt{(\gamma_1 - \gamma_2)^2 + 4\gamma_1\gamma_2\cos^2(\nu)}$$

$$\cos(2\nu_{EQ}) = \frac{\gamma_1/\gamma_2 + \cos(2\nu)}{\sqrt{(\gamma_1/\gamma_2 - 1)^2 + 4(\gamma_1/\gamma_2)\cos^2(\nu)}}$$

Note, that the resonant frequencies of the cavity just depend γ_{EQ} because of the splitting caused by birefringence. The two resonant frequencies are separated by γ_{EQ}. The resonant condition for one of the input polarizations is:

$$2n\pi = \omega_{res}\frac{2L}{c} + \gamma_{EQ}$$

Giving dependence of resonant frequency on this angle, and hence on ν:

$$\begin{aligned}\omega_{res} &= \frac{c}{2L}(2n\pi - \gamma_{EQ}) \\ &= \frac{c}{2L}(2n\pi - \sqrt{(\gamma_1 - \gamma_2)^2 + 4\gamma_1\gamma_2\cos^2(\nu)})\end{aligned}$$

To calculate effect of changes in ν to ω_{res}, let's calculate the first order derivative:

$$\frac{d\omega_{res}}{d\nu} = \frac{c}{L}\frac{\gamma_1\gamma_2\sin(2\nu)}{\sqrt{(\gamma_1 - \gamma_2)^2 + 4\gamma_1\gamma_2\cos^2(\nu))}}$$

Therefore, for a fixed ν_0 around which the vibrations occur due to seismic noise causing it to fluctuate by Φ as calculated above, coupling of seismic noise to resonance frequency noise due to birefringence of the mirrors is (in units of Hzs^2/m:

$$F_{birefringence-seismic} = \frac{c}{L}\frac{\gamma_1\gamma_2\sin(2\nu_0)}{2\pi\sqrt{(\gamma_1 - \gamma_2)^2 + 4\gamma_1\gamma_2\cos^2(\nu_0))}}\frac{2R_o M\delta\alpha L_A}{G\pi(R_o^4 - R_i^4)}$$

However, I believe in our cavities, the two mirrors were bonded close to making a $\nu_0 = \pi/2$ angle with each other which would make this coupling zero to first order. If we assume we made an error in aligning the mirrors by an angle $\delta\nu$, then the coupling would be ($\nu_0 = \pi/2 + \delta\nu$):

$$\begin{aligned}F_{birefringence-seismic} &= -\frac{c}{L}\frac{\gamma_1\gamma_2\sin(2\delta\nu)}{2\pi\sqrt{(\gamma_1 - \gamma_2)^2 + 4\gamma_1\gamma_2\cos^2(\delta\nu))}}\frac{2R_o M\delta\alpha L_A}{G\pi(R_o^4 - R_i^4)} \\ &\approx -\frac{c}{L}\frac{\gamma_1\gamma_2\delta\nu}{\pi|\gamma_1 - \gamma_2|}\frac{2R_o M\delta\alpha L_A}{G\pi(R_o^4 - R_i^4)}\end{aligned}$$

Approximations in current calculation

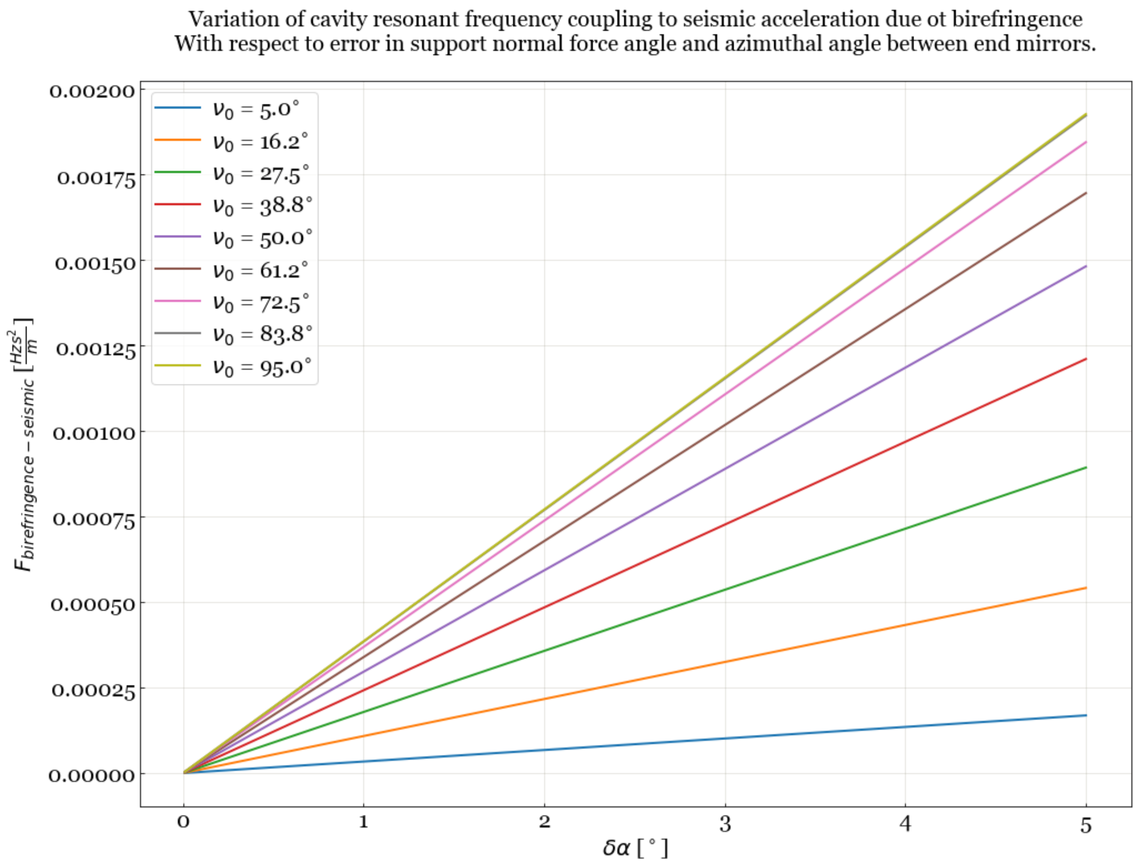
- Assuming both mirrors have same phase anisotropy $gamma$.
- We'll not assume ν_0 is 90° but we will calculate the seismic coupling for a few values of ν_0.

Then:

$$F_{birefringence-seismic} = \frac{c}{2\pi L}\gamma\sin(\nu_0)\frac{2R_o M\delta\alpha L_A}{G\pi(R_o^4 - R_i^4)}$$

In [7]:
```
# From Tenfold reduction of Brownian noise in optical interferometry supplementary i
# Difference in refractive index of AlGaAs/GaAs coating is roughly 1e-3
Dn = 1e-3                       # Birefringence, difference in refractive index
Lpen = 163e-9                   # Average penetration depth of light
k = 2*np.pi/1064e-9             # Wavenumber of laser
gamma = k*2*Lpen*Dn             # Phase anisotropy of mirror (assumed ot be same for b
nu_o = np.linspace(5, 95, 9)
Fbir_ses = np.zeros((len(nu_o), len(da)))
for ii, nu0 in enumerate(nu_o):
    Fbir_ses[ii, :] = (c/(2*np.pi*L))*gamma*np.sin(nu0*np.pi/180)*Phiar
fig = plt.figure()
ax = fig.gca()
for ii, nu0 in enumerate(nu_o):
    ax.plot(da, Fbir_ses[ii, :], label=r'$\nu_0$ = '+'{:.1f}'.format(nu0)+r'$^{\circ
ax.legend()
ax.set_title('Variation of cavity resonant frequency coupling to seismic acceleratio
             +'With respect to error in support normal force angle and azimuthal ang
ax.set_xlabel(r'$\delta \alpha$ [$^\circ$]')
ax.set_ylabel(r'$F_{birefringence-seismic}$ [$\frac{Hz s^2}{m}$]')
```

Out[7]:
```
Text(0, 0.5, '$F_{birefringence-seismic}$ [$\\frac{Hz s^2}{m}$]')
```



In [8]:
```
# From Tenfold reduction of Brownian noise in optical interferometry supplementary i
# Difference in refractive index of AlGaAs/GaAs coating is roughly 1e-3
Dn = 1e-3                      # Birefringence, difference in refractive index
Lpen = 163e-9                  # Average penetration depth of light
k = 2*np.pi/1064e-9            # Wavenumber of laser
gamma = k*2*Lpen*Dn            # Phase anisotropy of mirror (assumed ot be same for b
nu_o = np.linspace(5, 95, 9)
Fbir_ses = np.zeros((len(nu_o), len(da)))
for ii, nu0 in enumerate(nu_o):
    Fbir_ses[ii, :] = (1/(np.pi))*gamma*np.sin(nu0*np.pi/180)*Phiar
fig = plt.figure()
ax = fig.gca()
for ii, nu0 in enumerate(nu_o):
    ax.plot(da, Fbir_ses[ii, :], label=r'$\nu_0$ = '+'{:.1f}'.format(nu0)+r'$^{\circ
ax.legend()
ax.set_title('Variation of cavity strain coupling to seismic acceleration due ot bir
             +'With respect to error in support normal force angle and azimuthal ang
ax.set_xlabel(r'$\delta \alpha$ [$^\circ$]')
ax.set_ylabel(r'$S_{birefringence-seismic}$ [$\frac{s^2}{m}$]')
```

Out[8]:
```
Text(0, 0.5, '$S_{birefringence-seismic}$ [$\\frac{s^2}{m}$]')
```