The sense in which ϕ_c is a "coating loss angle"

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In this document we explain how the Nakagawa[1] loss angle ϕ_c, which is computed under the (generally fictitious) assumption that the coating and substrate have identical material parameters, is related to the true physical loss angle(s) of the coating materials. We thereby justify the statement made by Chalermsongsak et al.[2] that the Nakagawa loss angle is not an intrinsic material parameter, but rather a figure of merit for the performance of a particular coating.

1 The general Hong formalism for multilayer coatings

A general expression for coating Brownian noise is given by Hong et al. (eq. 94):[3]

$$S_x^{(\mathrm{cBr})}(f) = \sum_j \left[q_j^{(K)} S_j^{(K)}(f) + q_j^{(\mu)} S_j^{(\mu)}(f) \right]. \tag{1}$$

Here $S_j^{(K)}(f)$ and $S_j^{(\mu)}(f)$ are the noises due to the Brownian-driven fluctuation of the j^{th} layer of the coating, due either to bulk (K) or shear (μ) fluctuation, respectively (Hong eq. 96):

$$S_j^{(K,\mu)}(f) = \frac{4k_{\mathrm{B}}T\lambda}{3\pi^2 f n_j} \frac{(1+\sigma_j)(1-2\sigma_j)}{(1-\sigma_j)^2 w^2 E_j} \phi_j^{(K,\mu)}. \tag{2}$$

The dimensionless coefficients $q_j^{(K)}$ and $q_j^{(\mu)}$ involve the elastic and electromagnetic properties of the layer, and depend on the amount of light penetration into the layer.

2 Hong result for zero light penetration

For coatings made from quarter-wave layers of silica and tantala, the light penetration is only significant in the first few pairs of silica and tantala (Hong, fig. 7). Then to good approximation, we therefore may say there are only four coefficients—$q_{\mathrm{L}}^{(K)}$ and $q_{\mathrm{L}}^{(\mu)}$ for the low-index layer (silica), and $q_{\mathrm{H}}^{(K)}$ and $q_{\mathrm{H}}^{(\mu)}$ for the high-index layer (tantala)—and they depend only on the elastic moduli and indices of refraction of the quarter-wave layers. Under these assumptions, the expression for coating Brownian noise becomes

$$\begin{aligned} S_x^{(\mathrm{cBr})}(f) = {} & \frac{4k_{\mathrm{B}}T\lambda}{3\pi^2 f w^2} \\ & \times \left\{ \frac{N_{\mathrm{L}}}{n_{\mathrm{L}}} \frac{(1+\sigma_{\mathrm{L}})(1-2\sigma_{\mathrm{L}})}{(1-\sigma_{\mathrm{L}})^2 E_{\mathrm{L}}} \left[q_{\mathrm{L}}^{(K)}\phi_{\mathrm{L}}^{(K)} + q_{\mathrm{L}}^{(\mu)}\phi_{\mathrm{L}}^{(\mu)} \right] \right. \\ & \left. + \frac{N_{\mathrm{H}}}{n_{\mathrm{H}}} \frac{(1+\sigma_{\mathrm{H}})(1-2\sigma_{\mathrm{H}})}{(1-\sigma_{\mathrm{H}})^2 E_{\mathrm{H}}} \left[q_{\mathrm{H}}^{(K)}\phi_{\mathrm{H}}^{(K)} + q_{\mathrm{H}}^{(\mu)}\phi_{\mathrm{H}}^{(\mu)} \right] \right\} \end{aligned} \tag{3}$$

with

$$q_{\mathrm{X}}^{(K)} = \frac{d_{\mathrm{X}} n_{\mathrm{X}}}{\lambda} \frac{(1+\sigma_{\mathrm{X}})}{2} \left[1 + \frac{(1+\sigma_{\mathrm{s}})(1-2\sigma_{\mathrm{s}})}{(1+\sigma_{\mathrm{X}})} \frac{E_{\mathrm{X}}}{E_{\mathrm{s}}} \right]^2 \tag{4}$$

$$\begin{aligned} q_{\mathrm{X}}^{(\mu)} = {} & \frac{d_{\mathrm{X}} n_{\mathrm{X}}}{\lambda} (1-2\sigma_{\mathrm{X}}) \left\{ \left[1 - \frac{1}{2} \frac{(1+\sigma_{\mathrm{s}})(1-2\sigma_{\mathrm{s}})}{(1-2\sigma_{\mathrm{X}})} \frac{E_{\mathrm{X}}}{E_{\mathrm{s}}} \right]^2 \right. \\ & \left. + \frac{3}{4} \left[\frac{(1-\sigma_{\mathrm{X}})}{(1+\sigma_{\mathrm{X}})} \frac{(1+\sigma_{\mathrm{s}})(1-2\sigma_{\mathrm{s}})}{(1-2\sigma_{\mathrm{X}})} \frac{E_{\mathrm{X}}}{E_{\mathrm{s}}} \right]^2 \right\} \end{aligned} \tag{5}$$

for X = L, H. For these expressions we use eq. 94 and table I from Hong et al., with the light penetration parameter $\epsilon_j(z)$ (Hong eq. 25) set to zero.

3 Hong result under the assumption of equal bulk and shear loss angles

Currently, the bulk and shear loss angles $\phi^{(K)}$ and $\phi^{(\mu)}$ are not known independently, as experiments performed so far (including this one) are sensitive only some linear combination of the two. Keeping in mind that there is no reason to suppose that the bulk and shear loss angles are equal, we write $\phi_{\mathrm{L}}^{(K)} = \phi_{\mathrm{L}}^{(\mu)} \equiv \phi_{\mathrm{L}}$ and $\phi_{\mathrm{H}}^{(K)} = \phi_{\mathrm{H}}^{(\mu)} \equiv \phi_{\mathrm{H}}$. To facilitate comparison with a slightly simplified formula described below, we write

$$S_x^{(\mathrm{cBr})}(f) = \frac{4k_{\mathrm{B}}T}{\pi^2 f w^2} \left(\Xi_{\mathrm{L}} N_{\mathrm{L}} d_{\mathrm{L}} \phi_{\mathrm{L}} + \Xi_{\mathrm{H}} N_{\mathrm{H}} d_{\mathrm{H}} \phi_{\mathrm{H}} \right) \tag{6}$$

with

$$\Xi_{\mathrm{X}} = \frac{\lambda}{3n_{\mathrm{X}}} \frac{(1+\sigma_{\mathrm{X}})(1-2\sigma_{\mathrm{X}})}{(1-\sigma_{\mathrm{X}})^2 E_{\mathrm{X}}} \frac{\left[q_{\mathrm{X}}^{(K)} + q_{\mathrm{X}}^{(\mu)} \right]}{d_{\mathrm{X}}} \tag{7}$$

for X = L, H.

4 Comparison with the Nakagawa formula

The Nakagawa formula is derived under the assumption that the coating consists of a single, isotropic layer with loss an-

[1] "Thermal noise in half-infinite mirrors with nonuniform loss", *Phys Rev D 65*, 102001 (2001).

[2] "Broadband measurement of coating thermal noise in rigid Fabry–Pérot cavities", *Metrologia 52*, 17 (2015).

[3] "Brownian thermal noise in multilayer coated mirrors", *Phys. Rev. D 87*, 082001 (2013).

gle ϕ_c and with material parameters equal to the substrate's material parameters.

In the limit that $E_s = E_L = E_H \equiv E$, $\sigma_s = \sigma_L = \sigma_H \equiv \sigma$, and $\phi_H = \phi_L \equiv \phi_c$, we find

$$\Xi = \frac{(1+\sigma)(1-2\sigma)}{E} \tag{8}$$

and, since $N_L d_L \phi_c + N_H d_H \phi_c = d\phi_c$, we find

$$S_x^{(cBr)}(f) = \frac{4k_B T}{\pi^2 f w^2} \frac{\left(1-\sigma-2\sigma^2\right)d}{E} \phi_c, \tag{9}$$

which is the formula found previously by Nakagawa et al. (eq. 18) and Harry et al. (eq. 22).

Comparing this equation with eq. 6, we see that we can equate ϕ_c with a linear combination of ϕ_L and ϕ_H via

$$\frac{(1+\sigma)(1-2\sigma)d}{E} \phi_c = \Xi_L N_L d_L \phi_L + \Xi_H N_H d_H \phi_H, \tag{10}$$

with Ξ_L and Ξ_H given in eq. 7, and where E and σ are the substrate parameters. This equation show that, in general, ϕ_c differs from both ϕ_L and ϕ_H, and depends not only on the material parameters of the substrate and coating materials, but also on the geometry of the coating.