Transfer Function of Damped Harmonic Oscillator

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We are looking for a transfer function to convert our Force acting on the pendulum to the response of the pendulum. We are looking for

$$\frac{x(\omega)}{F(\omega)} = \boxed{},$$

where $x(\omega)$ is the movement of the pendulum and $F(\omega)$ is the force applied to the pendulum.

Given a driving force of $F = F_0 e^{i\omega t}$ the equation of motion for a driven damped harmonic oscillator is

$$0 = m\ddot{x} + b\dot{x} + kx + F_0 e^{i\omega t},$$

The solution to this differential equation is of the form $x = x_0 e^{i\omega t}$. When substituted in we get

$$\begin{aligned} 0 &= -m\omega^2 x_0 e^{i\omega t} + ib\omega x_0 e^{i\omega t} + kx_0 e^{i\omega t} + F_0 e^{i\omega t} \\ &= -m\omega^2 x_0 + ib\omega x_0 + kx_0 + F_0. \end{aligned}$$

Substituting in $b = m\omega_0/Q$, where Q is the quality factor and $k = m\omega_0^2$, we get

$$0 = -m\omega^2 x_0 + \frac{im\omega_0\omega}{Q} x_0 + m\omega_0^2 x_0 + F_0.$$

Now that we are living in frequency space we can Rearrange this equation to get our final answer:

$$\boxed{\frac{x(\omega)}{F(\omega)} = \frac{1}{m\omega_0^2 - m\omega^2 + im\frac{\omega_0\omega}{Q}}}$$

Checking units gives us:

$$\begin{aligned} \frac{[\mathrm{m}]}{[\mathrm{Kg}][\frac{\mathrm{m}}{\mathrm{s}^2}]} &= \frac{[1]}{[\mathrm{Kg}][\frac{1}{\mathrm{s}^2}] + [\mathrm{Kg}][\frac{1}{\mathrm{s}^2}] + [\mathrm{Kg}][\frac{1}{\mathrm{s}}][\frac{1}{\mathrm{s}}]} \\ \frac{[\mathrm{s}^2]}{[\mathrm{Kg}]} &= \frac{[1]}{[\mathrm{Kg}][\frac{1}{\mathrm{s}^2}]} \\ &= \frac{[\mathrm{s}^2]}{[\mathrm{Kg}]} \quad \checkmark \end{aligned}$$