Mode coupling of two astigmatic gaussian beams

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1 Introduction

In this note, amplitude and power couplings of two astigmatic (0, 0)-th order gaussian modes are calculated.

2 Astigmatic Hermite gaussian beams

An electric field of the astigmatic $(n,\, m)$-th Hermite-gaussian beam propagating to the z direction is expressed as the following form [1]:

$$\begin{aligned} U_{nm}(x,\, y,\, z) &= u_n(x,\, z)\, u_m(y,\, z) \\ &= \left(\frac{2}{\pi}\right)^{1/4} \left(\frac{1}{2^n n!\, \omega_{0x}}\right)^{1/2} \left(\frac{\tilde{q}_{0x}}{\tilde{q}_x(z)}\right)^{1/2} \left[\frac{\tilde{q}_{0x}}{\tilde{q}^*_{0x}} \frac{\tilde{q}^*(z)}{\tilde{q}(z)}\right]^{n/2} H_n\left(\frac{\sqrt{2}x}{\omega_x(z)}\right) \exp\left[-\frac{\mathrm{i}k}{\tilde{q}_x(z)}\frac{x^2}{2}\right] \\ &\times \left(\frac{2}{\pi}\right)^{1/4} \left(\frac{1}{2^m m!\, \omega_{0y}}\right)^{1/2} \left(\frac{\tilde{q}_{0y}}{\tilde{q}_y(z)}\right)^{1/2} \left[\frac{\tilde{q}_{0y}}{\tilde{q}^*_{0y}} \frac{\tilde{q}^*(z)}{\tilde{q}(z)}\right]^{m/2} H_m\left(\frac{\sqrt{2}y}{\omega_y(z)}\right) \exp\left[-\frac{\mathrm{i}k}{\tilde{q}_y(z)}\frac{y^2}{2}\right] \quad . \end{aligned} \tag{1}$$

Here, $\tilde{q}_x(z)$ is the q-parameter in the horizntal direction, being defined by $\tilde{q}_x(z) \equiv z - z_{0x} + \tilde{q}_{0x}$, where $\tilde{q}_{0x} = \mathrm{i}z_{Rx}$. z_{Rx} is the Rayleigh range, being defined by $z_{Rx} = \pi\omega_{0x}^2/\lambda$, where ω_{0x} is the waist size in the horizontal direction. Same definitions for the vertical direction by replacing x to y. k is the wave number and defined by $k \equiv 2\pi/\lambda$, where λ is the wavelength of the laser beam. The symbols *tilde* (˜) express complex numbers.

Since we assume $n = m = 0$, the expression for the beam is simplified to the following form:

$$U_{00}(x,\, y,\, z) = \left(\frac{k}{\pi z_{Rx}}\right)^{1/4} \sqrt{\frac{\mathrm{i}z_{Rx}}{\tilde{q}_x(z)}} \exp\left(-\frac{\mathrm{i}k}{\tilde{q}_x(z)}\frac{x^2}{2}\right) \times \left(\frac{k}{\pi z_{Ry}}\right)^{1/4} \sqrt{\frac{\mathrm{i}z_{Ry}}{\tilde{q}_y(z)}} \exp\left(-\frac{\mathrm{i}k}{\tilde{q}_y(z)}\frac{y^2}{2}\right) \quad . \tag{2}$$

By writing q parameters more explicitly, we obtain

$$\begin{aligned} &\varphi(x,\, y,\, z,\, z_{Rx},\, z_{0x},\, z_{Ry},\, z_{0y}) \equiv U_{00}(x,\, y,\, z) \\ &= \sqrt{\frac{k}{\pi\sqrt{z_{Rx} z_{Ry}}}} \sqrt{\frac{\mathrm{i}z_{Rx}}{z - z_{0x} + \mathrm{i}z_{Rx}}} \exp\left(\frac{-\mathrm{i}k}{z - z_{0x} + \mathrm{i}z_{Rx}}\frac{x^2}{2}\right) \sqrt{\frac{\mathrm{i}z_{Ry}}{z - z_{0y} + \mathrm{i}z_{Ry}}} \exp\left(\frac{-\mathrm{i}k}{z - z_{0y} + \mathrm{i}z_{Ry}}\frac{y^2}{2}\right) \quad , \end{aligned} \tag{3}$$

z_{Rx} and z_{Ry} are the Rayleigh ranges of the beam in the horizontal and vertical directions, respectively. z_{0x} and z_{0y} are the waist positions in the horizontal and vertical directions, respectively.

3 Coupling of two beams

The amplitude coupling (i.e. the expansion coefficient) of the two such beams can be expressed as the following integration:

$$C = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(x,\, y,\, z,\, z_{Rx1},\, z_{1x},\, z_{Ry1},\, z_{1y})\, \varphi^*(x,\, y,\, z,\, z_{Rx2},\, z_{2x},\, z_{Ry2},\, z_{2y})\, \mathrm{d}x\, \mathrm{d}y \tag{4}$$

This C is the amplitude coupling and a complex number in general. C can be simplified by using the relationship $\int_{-\infty}^{\infty} \mathrm{e}^{-Ax^2} \mathrm{d}x = \sqrt{\pi/A}$ for $\mathrm{Re}(A) > 0$.

$$C = 2\sqrt{\frac{\sqrt{z_{Rx1}\, z_{Rx2}\, z_{Ry1}\, z_{Ry2}}}{[\mathrm{i}(z_{1x} - z_{2x}) + (z_{Rx1} + z_{Rx2})][\mathrm{i}(z_{1y} - z_{2y}) + (z_{Ry1} + z_{Ry2})]}} \tag{5}$$

Note that this coupling is independent z.

The mode matching (or mode overlapping) can be obtained by the power coupling. This can be obtained by

$$|C|^2 = 4\sqrt{\frac{z_{Rx1}\, z_{Rx2}\, z_{Ry1}\, z_{Ry2}}{[(z_{1x} - z_{2x})^2 + (z_{Rx1} + z_{Rx2})^2][(z_{1y} - z_{2y})^2 + (z_{Ry1} + z_{Ry2})^2]}} \ . \tag{6}$$

In the special case that the two modes are not astigmatic (i.e. $z_{Rx1} = z_{Ry1} = z_{R1}$, $z_{Rx2} = z_{Ry2} = z_{R2}$, $z_{1x} = z_{1y} = z_1$, $z_{2x} = z_{2y} = z_2$), C and $|C|^2$ are expressed as the followings:

$$C = 2\frac{\sqrt{z_{R1}\, z_{R2}}}{\mathrm{i}(z_1 - z_2) + (z_{R1} + z_{R2})} \tag{7}$$

$$|C|^2 = 4\frac{z_{R1}\, z_{R2}}{(z_1 - z_2)^2 + (z_{R1} + z_{R2})^2} \ . \tag{8}$$

References

[1] Eq.(54), Sec. 16.4, A. E Siegman, *Lasers*, University Science Books (1986).