# 1 ETM Tilt

Cavity mode: $u_0(x) = \exp\left(-\frac{x^2}{{w_0}^2}\right)$
Incident beam: $u(x) = \exp\left(-\frac{(x-\Delta x)^2}{{w_0}^2}\right)$ with $\Delta x = R\sin\alpha \approx R\alpha$

Overlap $\Gamma_{\mathrm{ETM}}$:
$\Gamma_{\mathrm{ETM}} = \left|\int u_0(x)u(x)\mathrm{d}x\right| = \left|\int \exp\left(-\frac{2x^2-2x\Delta x+\Delta x^2}{{w_0}^2}\right)\mathrm{d}x\right| = \left|\int \exp\left(-\frac{2x^2-2x\Delta x+\frac{\Delta x^2}{2}}{{w_0}^2}\right)\exp\left(-\frac{\Delta x^2}{2{w_0}^2}\right)\mathrm{d}x\right|$
$= \exp\left(-\frac{\Delta x^2}{2{w_0}^2}\right)\left|\int \exp\left(-\frac{2(x-\frac{\Delta x}{2})^2}{{w_0}^2}\right)\mathrm{d}x\right| \overset{y=x-\frac{\Delta x}{2}}{=} \exp\left(-\frac{\Delta x^2}{2{w_0}^2}\right)\left|\int \exp\left(-\frac{2y^2}{{w_0}^2}\right)\mathrm{d}y\right|$
$\Rightarrow$ Overlap reduces as $\exp\left(-\frac{\Delta x^2}{2{w_0}^2}\right)$
$\Rightarrow$ Transmission changes with $\exp\left(-\frac{\Delta x^2}{{w_0}^2}\right)$

Fit data for transmitted power versus induced tilt $v$ to $T(v) = T_0\exp\left(-\frac{(v-v_0)^2}{{w_v}^2}\right) + \text{offset}$
$\Rightarrow \frac{R\alpha}{w_0} = \frac{\Delta x}{w_0} = \frac{v-v_0}{w_v} \Rightarrow \alpha = \frac{1}{R}\frac{w_0}{w_v}(v-v_0)$

Fit data for OpLev signal $y$ versus induced tilt $v$ to $y(v) = mv + b$
$\Rightarrow v = \frac{y-b}{m}$

To get the differential calibration we can neglect $v_0$ and $b$, and obtain

$$\alpha_{\mathrm{ETM}} = \frac{w_0/w_v}{mR} \times y$$

# 2 ITM Tilt

Cavity mode: $u_0(x) = \exp\left(-\frac{x^2}{{w_0}^2}\right)$
Incident beam: $u(x) \approx \exp\left(-\frac{(x-\Delta x)^2}{{w_0}^2} + ikx\alpha\right)$ with $\Delta x = (R-L)\tan\alpha \approx (R-L)\alpha$

Overlap $\Gamma_{\mathrm{ITM}}$:
$\Gamma_{\mathrm{ITM}} = \left|\int u_0(x)u(x)\mathrm{d}x\right| = \left|\int \exp\left(-\frac{2x^2-2x\Delta x+\Delta x^2-ikx\alpha{w_0}^2}{{w_0}^2}\right)\mathrm{d}x\right|$
$= \left|\int \exp\left(-\frac{2x^2-2x\left[\Delta x+i\frac{k\alpha{w_0}^2}{2}\right]+\frac{[\Delta x+i\frac{k\alpha{w_0}^2}{2}]^2}{2}}{{w_0}^2}\right)\exp\left(-\frac{\Delta x^2}{2{w_0}^2}\right)\exp\left(i\frac{k\alpha\Delta x}{2}\right)\exp\left(-\frac{k^2\alpha^2{w_0}^2}{8}\right)\mathrm{d}x\right|$
$= \exp\left(-\frac{[(R-L)^2+\frac{k^2{w_0}^4}{4}]\alpha^2}{2{w_0}^2}\right)\left|\int \exp\left(-\frac{2\left(x-\frac{\Delta x}{2}-i\frac{k\alpha{w_0}^2}{2}\right)^2}{{w_0}^2}\right)\mathrm{d}x\right|$

$$\overset{y=x-\frac{\Delta x}{2}}{=} \exp\left(-\frac{[(R-L)^2+\frac{k^2 {w_0}^4}{4}]\alpha^2}{2{w_0}^2}\right) \left| \int \exp\left(-\frac{2\left(y-i\frac{k\alpha {w_0}^2}{2}\right)^2}{{w_0}^2}\right) \mathrm{d}y \right|$$

$$\overset{z=y-i\frac{k\alpha {w_0}^2}{2}}{=} \exp\left(-\frac{[(R-L)^2+\frac{k^2 {w_0}^4}{4}]\alpha^2}{2{w_0}^2}\right) \left| \int \exp\left(-\frac{2z^2}{{w_0}^2}\right) \mathrm{d}z \right| \text{ (requires contour integration)}$$

$\Rightarrow$ Overlap reduces as $\exp\left(-\frac{[(R-L)^2+\frac{k^2 {w_0}^4}{4}]}{2{w_0}^2}\alpha^2\right)$

$\Rightarrow$ Transmission changes as $\exp\left(-\frac{[(R-L)^2+\frac{k^2 {w_0}^4}{4}]}{{w_0}^2}\alpha^2\right)$

Fit data for transmitted power versus induced tilt $v$ to $T(v) = T_0 \exp\left(-\frac{(v-v_0)^2}{{w_v}^2}\right) + \text{offset}$

$\Rightarrow \frac{\sqrt{(R-L)^2+\frac{k^2 {w_0}^4}{4}}}{w_0}\alpha = \frac{v-v_0}{w_v} \Rightarrow \alpha = \frac{1}{\sqrt{(R-L)^2+\frac{k^2 {w_0}^4}{4}}}\frac{w_0}{w_v}(v-v_0)$

Fit data for OpLev signal $y$ versus induced tilt $v$ to $y(v) = mv + b$
$\Rightarrow v = \frac{y-b}{m}$

To get the differential calibration we can neglect $v_0$ and $b$, and obtain

$$\alpha_{\mathrm{ITM}} = \frac{w_0/w_v}{m\sqrt{(R-L)^2+\frac{k^2 {w_0}^4}{4}}} \times y$$