Stewart platform design study

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The Stewart platform is a six degrees of freedom (DOF) positioning device consisting of two rigid plates joined by six extendable legs. Possible applications to gravitational wave detection include characterization of suspensions and seismic isolation. In this design study, I consider a tabletop platform designed to support a load comparable to the small optic suspension (SOS), with a linear and angular travel range comparable with the design requirements (LIGO-T950011) of the SOS.

The design entails choosing:

- the actuator technology: piezoelectric, piezoelectric stepper, inductive, pneumatic, hydraulic, or electric stepper motor
- the sensing technolog(ies): strapdown rate gyros and accelerometers, linear encoders, optical levers, interferometry
- the control strategy: analog or digital, PID control of leg lengths, extended Kalman filter and linear quadratic regulator
- joints: ball joint, universal coupling, or flexure
- material of the plates
- geometry: optimal plate dimensions and joint locations

In terms of quantities in Table 1, the linear travel imposes a requirement that the actuators collectively be capable of delivering a peak force of at least $\frac{1}{2}m\Delta x_{\mathrm{pp}}(2\pi f)^2$, and the angular travel requires that the actuators be collectively capable of delivering a peak torque of at least $\frac{1}{2}I\Delta\phi_{\mathrm{pp}}(2\pi f)^2$. For the mass, moment of inertia, travel limits, and bandwidth in the table, the net force requirement is about 2.32 kN and the net torque requirement is about 343 N m. The ratio of the peak net torque to the peak net force, 0.15 m, provides an initial guess about the dimensions of the plates.

1 Dynamics and optimum geometry

A crude model of the dynamics of the Stewart platform will be needed to select the optimal geometrical configuration.

The kinematic state of the system is fully described by the vector

$$\mathbf{q} = [u \quad v \quad w \quad \alpha \quad \beta \quad \gamma \quad \dot{u} \quad \dot{v} \quad \dot{w} \quad \omega_1 \quad \omega_2 \quad \omega_3]^\mathsf{T}$$

quantity	value	note
linear travel (peak to peak) Δx_{pp}	40 μm	as listed in LIGO-T950011
angular travel (peak to peak) $\Delta\phi_{\mathrm{pp}}$	3 mrad	as listed in LIGO-T950011
mass of payload m	11.75 kg	computed from measured mass of unloaded SOS, estimated mass of orientation sensor and motors (OSEMs), and specified mass of optic in LIGO-T970135
moment of inertia of payload I	0.0232 kg m^2	guess
bandwidth f	500 Hz	guess
net force (peak)	2.32 kN	
net torque (peak)	343 N m	
net torque (peak) / net force (peak)	0.15 m	
peak actuator force	2.04 kN	computed in section 1.1
radius of top platform r_a	15 cm	computed in section 1.1
radius of bottom platform r_b	30 cm	computed in section 1.1
height of platform	26 cm	computed in section 1.1

Table 1: Design parameters for tabletop platform.

where $\mathbf{x} = [u \quad v \quad w]$ is the Cartesian displacement vector of the top platform, α, β, and γ are the Euler angles describing the rotation of the platform, $\dot{u}$, $\dot{v}$, and $\dot{w}$ are the components of the velocity of the plate, and ω_1, ω_2, and ω_3 are the components of the angular velocity vector.

The most general Stewart platform has six joints arbitrarily attached in space to a rigid, but not necessarily planar, base. The Cartesian coordinates of these six joints in the inertial frame of the apparatus are $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_6$. The (not necessary planar) top platform also has six joints attached to it. In the "home" state, $u = v = w = \alpha = \beta = \gamma = 0$, the positions of the top joints in the inertial frame are $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_6$. Under a displacement and a rotation of the top plate, the positions of the top joints become

$$R\mathbf{a}_i + \mathbf{x}.$$

The leg vectors are defined as

$$\mathbf{L}_i = R\mathbf{a}_i - \mathbf{b}_i + \mathbf{x},$$

with the leg length $L_i = \|\mathbf{L}_i\|$, and the unit leg vectors $\hat{\mathbf{L}}_i = \mathbf{L}_i / L_i$.

The equation of motion of the platform (see, for example, Kang et al., 1996) is

$$M\dot{\mathbf{v}} + C\mathbf{v} + G = J^{\mathsf{T}}\mathbf{u}$$

where $\mathbf{v} = [\dot{u} \quad \dot{v} \quad \dot{w} \quad \omega_1 \quad \omega_2 \quad \omega_3]^{\mathsf{T}}$, M is the mass matrix, C describes the Coriolis and centrifugal forces, G is gravity, $\mathbf{u} = [u_1 \quad u_2 \quad \ldots \quad u_6]^{\mathsf{T}}$ is the vector of the forces applied at the leg joints, and J is a Jacobian matrix relating changes in leg length and changes in kinematic state, defined by

$$J \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \dot{L}_1 \\ \dot{L}_2 \\ \dot{L}_3 \\ \dot{L}_4 \\ \dot{L}_5 \\ \dot{L}_6 \end{bmatrix}.$$

The Jacobian for the Stewart platform is (Tsai, 1999)

$$J = \begin{bmatrix} \hat{\mathbf{L}}_1^{\mathsf{T}} & \left(\hat{\mathbf{L}}_1 \times (R\mathbf{a}_1)\right)^{\mathsf{T}} \\ \hat{\mathbf{L}}_2^{\mathsf{T}} & \left(\hat{\mathbf{L}}_2 \times (R\mathbf{a}_2)\right)^{\mathsf{T}} \\ \vdots & \vdots \\ \hat{\mathbf{L}}_6^{\mathsf{T}} & \left(\hat{\mathbf{L}}_6 \times (R\mathbf{a}_6)\right)^{\mathsf{T}} \end{bmatrix}.$$

Notice that the transpose of the first three columns forms an operator that maps the actuator forces to the net force on the payload,

$$\mathbf{F}_{\text{net}} = \left[\hat{\mathbf{L}}_1 \quad \hat{\mathbf{L}}_2 \quad \cdots \quad \hat{\mathbf{L}}_6\right]\mathbf{u},$$

and the transpose of the last three columns forms an operator that maps the actuator forces to the net torque on the payload,

$$\mathbf{N}_{\text{net}} = \left[\left(\hat{\mathbf{L}}_1 \times (R\mathbf{a}_1)\right) \quad \left(\hat{\mathbf{L}}_2 \times (R\mathbf{a}_2)\right) \quad \cdots \quad \left(\hat{\mathbf{L}}_6 \times (R\mathbf{a}_6)\right)\right]\mathbf{u}.$$

This suggests constructing figures of merit from a modified Jacobian J',

$$J' = \begin{bmatrix} \hat{\mathbf{L}}_1^\mathsf{T} & w^{-1}\left(\hat{\mathbf{L}}_1 \times (R\mathbf{a}_1)\right)^\mathsf{T} \\ \hat{\mathbf{L}}_2^\mathsf{T} & w^{-1}\left(\hat{\mathbf{L}}_2 \times (R\mathbf{a}_2)\right)^\mathsf{T} \\ \vdots & \vdots \\ \hat{\mathbf{L}}_6^\mathsf{T} & w^{-1}\left(\hat{\mathbf{L}}_6 \times (R\mathbf{a}_6)\right)^\mathsf{T} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & w^{-1}I \end{bmatrix} J,$$

where w is a design parameter with the dimension of distance that relates the torque and force requirements.

One possible choice of objective function is the absolute value of the determinant of the modified Jacobian evaluated at specified point in configuration space,

$$f(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_6, \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_6; \mathbf{x}) = |\det J'|.$$

The determinant relates the infinitesimal volume elements of force and torque to infinitesimal changes in the actuator forces through $\mathrm{d}^3\mathbf{F}_\mathrm{net}\,\mathrm{d}^3(\mathbf{N}_\mathrm{net}/w) = |\det J'|\,\mathrm{d}^6\mathbf{u}$. Unfortunately, the optimum of this objective function is insensitive to the torque-to-force weighting because, using the multiplicative property of determinants,

$$\det J' = w^{-3} \det J.$$

Stoughton and Arai (1993) propose another objective function, the *dexterity index*

$$f(\cdot) = V^* \left[\int_{V^*} \kappa(J')\,\mathrm{d}^6V\right]^{-1}.$$

d^6V the volume element in state space and V^* is a central subregion of interest in state space. $\kappa(J')$ is the condition number of the modified Jacobian, defined as

$$\kappa(J') = \|J'\|\|(J')^{-1}\| \geq 1,$$

where $\|\cdot\|$ denotes the Frobenius norm. The condition number is also the ratio of the maximum and minimum singular values, or the square of the ratio of the maximum and minimum eigenvalues. The condition number describes the isotropy of the force and torque authority.

1.1 Optimum design of a symmetric Stewart platform

The joint positions alone, $\mathbf{a}_i$ and $\mathbf{b}_i$, constitute twelve degrees of freedom. The dimension of the optimization problem can be greatly reduced by introducing symmetry. I adopt a parameterization similar to Davliakos and Papadopoulos (2008). The placement of the bottom joints starts with an equilateral triangle with its centroid at the origin and vertices a distance r_b from the origin. Two joints are placed near each of the vertices. Due to interference, two joints cannot exactly coincide, so each joint is displaced from the neighboring vertex of the triangle such that the joint, the origin, and the vertex form an angle α_b. Similarly, the top joints are placed near the vertices of another equilateral triangle, but rotated 60°, a distance r_a from the origin and displaced by an able α_a from the vertices. This symmetric configuration is depicted in Figure 1. Under this parameterization, the determinant of the modified Jacobian is

$$|\det J'| = \frac{27 r_a^3 r_b^3 z^3 \left[3\cos\left(\alpha_a+\alpha_b+\pi/6\right) - \sin 3(\alpha_a+\alpha_b)\right]}{2w^3 \left[r_a^2 + r_b^2 + z^2 - 2r_a r_b \sin\left(\alpha_a+\alpha_b+\pi/6\right)\right]^3}.$$

Clearly, the Jacobian determinant increases with the overall scale of the platform, so we can fix the overall scale by setting $r_a = 1$ or $r_b = 1$. If we set $r_a = 1$, the optimum is obtained when

$$r_b = \sqrt{1+z^2}, \quad \alpha_a + \alpha_b = \pm \arccos\left[\frac{1+\sqrt{3}z}{2\sqrt{1+z^2}}\right],$$

where the $+$ solution is taken if $z > \sqrt{3}$ and the $-$ solution is taken if $z < \sqrt{3}$. If $z = \sqrt{3}$, then the optimum is obtained when $r_b = 2$ and $\alpha_a + \alpha_b = 0$.

For this particular optimum, the actuator force requirement and the overall scale of the platform have to come from the net force and torque requirements. The pseudoinverse of the first three rows of J^{T} is an operator that transforms the net force into the actuator forces. This pseudoinverse is

$$\begin{bmatrix} 0 & -\frac{\sqrt{2}}{3} & \frac{1}{3\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} \\ 0 & \frac{\sqrt{2}}{3} & \frac{1}{3\sqrt{2}} \end{bmatrix}.$$

The greatest required force for actuator i will occur if the net force is along the direction given by row i of this matrix, so the maximum required force for each actuator is the peak force times the sum of the absolute values of the entries in each row. From this consideration, the maximum required actuator force is

$$F_{\max} = \left(\frac{2+\sqrt{3}}{3\sqrt{2}}\right) \cdot 2.32 \text{ kN} \approx 2.04 \text{ kN}.$$

The pseudoinverse of the last three rows of J^{T} is an oeprator that transforms the net torque into actuator forces. This pseudoinverse is

$$\frac{1}{r_a}\begin{bmatrix} 0 & \frac{\sqrt{2}}{3} & \frac{1}{3\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} \\ 0 & \frac{\sqrt{2}}{3} & -\frac{1}{3\sqrt{2}} \end{bmatrix}.$$

Similarly, the greatest required force for actuator i will occur if the net torque is along the direction given by row i of this matrix, so the maximum required force for each actuator is the peak torque times the sum of the absolute values of the entries in each row. This gives

$$F_{\max} = \frac{1}{r_a} \left(\frac{2+\sqrt{3}}{3\sqrt{2}} \right) \cdot 343 \text{ N m}.$$

Adopting the ratio of the net torque requirement to the net force requirement as the weighting $w = 0.15$, and substituting the maximum actuator force found above, we find that $r_a = 0.15$ m is sufficient to meet the torque requirement.

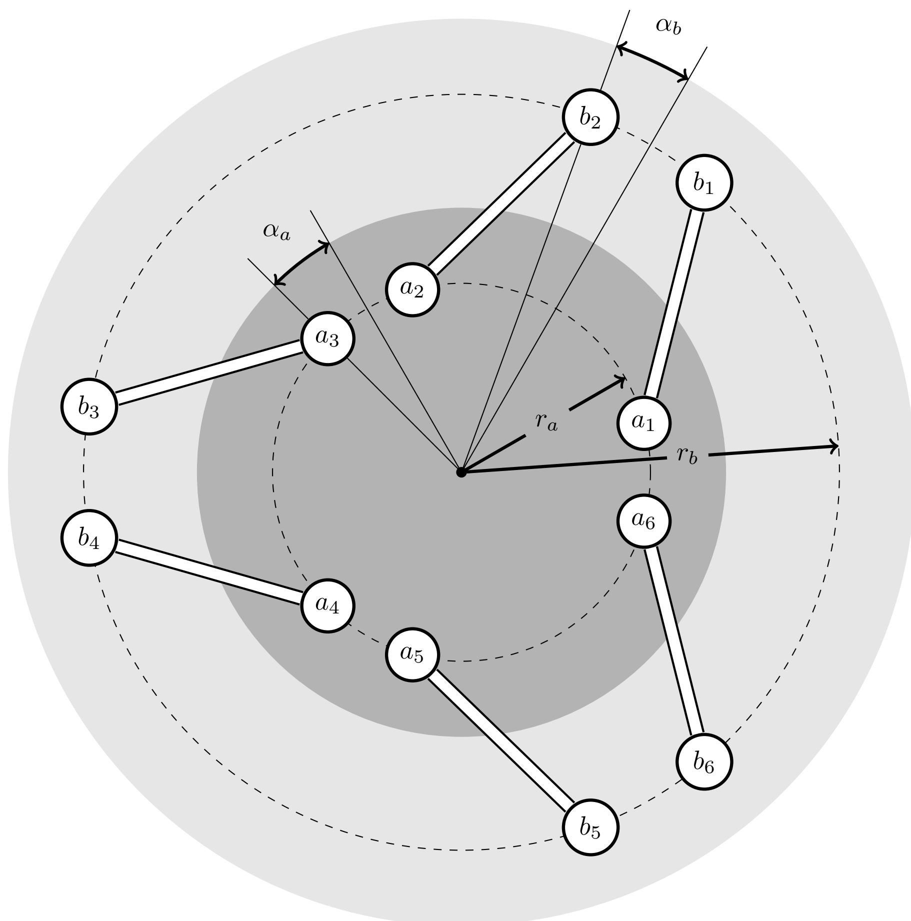

Figure 1: Plan view of a symmetric Stewart platform; visual presentation inspired by Davliakos and Papadopoulos (2008). The fixed positions of the bottom joints are denoted $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_6$ and the positions of the top joints in the home configuration are $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_6$. The bottom joints are a radius r_b from the center of the bottom plate and the top joints are a radius r_a from the center of the top plate. The half-angle separating neighboring joints on the bottom plate is α_b, and on the top plate, α_a.

$$\mathbf{a}_i = \begin{bmatrix}\cos\gamma_i^a & \sin\gamma_i^a & 0\end{bmatrix}^\mathsf{T} \qquad \gamma_i^a = \left\lfloor \frac{i}{2} \right\rfloor \frac{2\pi}{3} - (-1)^i \alpha_a$$

$$\mathbf{b}_i = \begin{bmatrix}\cos\gamma_i^b & \sin\gamma_i^b & 0\end{bmatrix}^\mathsf{T} \qquad \gamma_i^b = \left\lceil \frac{i}{2} \right\rceil \frac{2\pi}{3} + (-1)^i \alpha_b - \frac{\pi}{3}$$

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